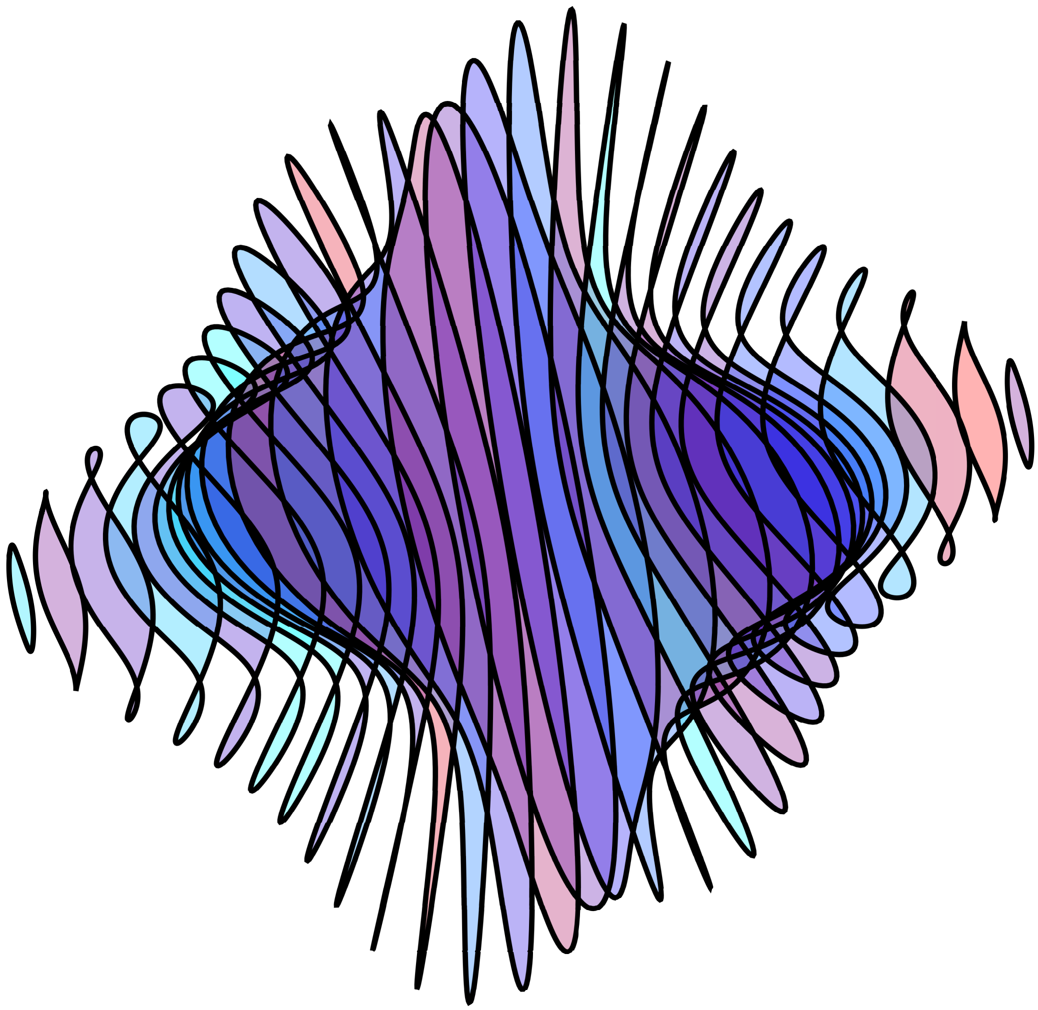
Multinomial Series Inversion and an Empirical Conjecture



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I. INTRODUCTION

Let function $F(x)$ equal to z in an *implicit equation.* Additionally, require that the function expands in a power series around $x = 0$, the *forward series*,

$$z = F_m(\mathbf{c}, x) = \sum_{i=0}^{\infty} c_i \, x^{m+i}, \tag{1}$$

where parameter m, the *leading power*, distinguishes between different cases. Assuming F_m invertible, introduce *functional inverse*

$$x = G_m(\mathbf{c}, z), \tag{2}$$

which satisfies the *inversion composition condition*,

$$z = F_m(G_m(z)), \tag{3}$$

with $\mathbf{c}$ dependence suppressed. When F_m and G_m satisfy condition Eq. 3, we say that G_m solves implicit equation $F_m(x) = z$.

The present note discusses an easily provable definition for G_m and subsequently proposes an improved form, as yet known to the author only as conjecture. The conjectural form appears to be correct for all $m > 0$. Relative to the simple result, the conjectural form is proven approximately true in many cases of brute force analysis.

II. SMALL AMPLITUDE LIMIT

As $x \longrightarrow 0$, all higher order terms of F_m approach zero more rapidly than the leading term x^m,

$$\lim_{x \longrightarrow 0} \left(z = F_m(c_0, x) = c_0 \, x^m \right). \tag{4}$$

The limit equation admits a trivial solution

$$\lim_{x \longrightarrow 0} \left(x = y = (z/c_0)^{1/m} \right). \tag{5}$$

By recursive, algebraic arguments, we then assume a solution in powers of y, the *reverse series*,

$$G_m(\mathbf{c}, z = c_0 \, y^m) = G'_m(\mathbf{c}, y) = y \, \sum_{i=0}^{\infty} g_{m,i}(\mathbf{c}) \, y^i, \tag{6}$$

such that

$$z = g_{m,0}\, c_0\, y^m = F_m(G'_m(y)), \tag{7}$$

with the extra requirement $g_{m,0} = 1$. According to Eq. 5, if c_0 equals to zero the inversion solution G_m immediately diverges to infinity. However, change of variables $x' = (c_0)^{1/m}\, x$ effectively sets $c_0 = 1$, a reduction that greatly simplifies the determination of G_m.

III. TRUNCATION CONVENTION

In the practice of explicit calculation, it helps to introduce a *truncation parameter*, call it n, which changes infinite summations to finite expressions with error terms

$$F_{m,n}(\mathbf{c}, x) = \left(\sum_{i=0}^{n} c_i\, x^{m+i} \right) + \mathcal{O}(x^{n+m+1}), \tag{8a}$$

$$G'_{m,n}(\mathbf{c}, y) = y \left(\sum_{i=0}^{n} g_{m,i}(\mathbf{c})\, y^i \right) + \mathcal{O}(y^{n+2}). \tag{8b}$$

Truncated series yield approximate satisfaction of the composition condition

$$H_{m,n}(y(z)) = F_{m,n}(G'_{m,n}(y)) = y^m + \mathcal{O}(y^{n+m+1}) = z + \mathcal{O}(z^{(n+m+1)/m}), \tag{9}$$

where again the convention requires $g_{m,0} = c_0 = 1$.

IV. DIRECT SOLUTION OF THE RECURSION EQUATIONS

To satisfy Eq. 9, the coefficient of each y^{m+i} in the expansion of $H_{m,n}$ must equal zero for $i \in [1, n]$. For each i this yields an equation linear in $g_{m,i}$, which is trivial to solve in terms only of the lesser $g_{m,k}$ with $k \in [0, i-1]$. Although $g_{m,0}$ always equals 1, the symbolic form is useful to include when counting powers and evaluating exponent conditions. The *recursion equations* are

$$y^{m+i} : 0 = \sum_{\Gamma_{m,i}} \gamma_{m,i}(\mathbf{v}), \tag{10}$$

and the summation occurs over the set

$$\Gamma_{m,i} = \left\{ \gamma_{m,i}(\mathbf{v}) = c_{j-m}\, [\mathbf{v}]\, \prod_{k=0}^{i} \left(g_{m,k}\right)^{v_k} : \left(\textstyle\sum_k v_k\, (k+1) = m + i \right) \right. \\ \left. \& \left(\textstyle\sum_k v_k = j \geq m \right) \right\}. \tag{11}$$

In set Γ multiple $[\mathbf{v}]$ is the multinomial coefficient

$$[\mathbf{v}] = \frac{\left(\sum_k v_k\right)!}{\prod_k v_k!}, \tag{12}$$

for the collection of integers $\mathbf{v} = \{v_k\}$, all greater than or equal to zero. The first condition of Eq. 11 ensures that the γ term attaches to y^{m+i}, while the second condition ensures that such a term occurs in the expansion of $c_{j-m}(G'_{m,n})^j$. In practice, the set of $\mathbf{v}$ satisfying the selection constraints depends quickly on a map from the integer partitions of $m+i$, where exponent v_k counts the repetitions of integer $k+1$.

The special term $g_{m,i}$ first occurs in the y^{m+i} summation when $\sum_l v_l = j = m$, with a prefactor $(c_0\,[1, m-1]) = m$. Moving this term to the left hand side, and multiplying both sides by $(-1/m)$ solves the system of equations.

In practice it is possible to program $G'_{m,n}(\mathbf{c}, y)$ using only simple, recursive functions. Working in Mathematica, the multinomial algorithm is fast relative to an output-equivalent algorithm, which uses the built-in mystery function `InverseSeries`. Timing tests on a personal computer, show that up to 2 seconds, the slow algorithm computes to $n = 10$. In the same time the fast algorithm computes at least to $n = 18$, sometimes to $n = 20$, depending on leading power m. Timing tests are depicted in Fig. 1, of section VII.

V. EXTRACTING COEFFICIENTS

According to the solved recursion equations with $c_0 = 1$, the $g_{m,i}$ with $i > 0$ must contain only those c_j with $j \in [1, i]$. Maximally, $g_{m,i}$ could then be a summation over every product

$$\widetilde{A}_i = \left\{ \widetilde{a}_i(\mathbf{w}, \mathbf{c}) = \prod_{w_l \in \mathbf{w}} c_{w_l} : \sum_l^{L(\mathbf{w})} w_l = i \right\}, \tag{13}$$

where $\mathbf{w} = \{w_l\}$ is a partition of i with all $w_l > 0$ and length $L(\mathbf{w})$. The function G_m then expands in terms of

$$g_{m,i}(\mathbf{c}) = \sum_{j=1}^{p(i)} a_{m,i,j}\, \widetilde{a}_i(\mathbf{w}_j, \mathbf{c}). \tag{14}$$

Index j introduces an arbitrary ordering of integer partitions.

Each $a_{m,i,j} \in \mathbb{Q}$ of Eq. 14 is a rational coefficient. Per m value, the $a_{m,i,j}$ coefficients constitute a *partition triangle*, so named because the row lengths equal the partition numbers.

A special case occurs for $m = 1$ where $a_{1,i,j} \in \mathbb{Z}$ as in A111785. Case $m = 2$ requires gerrymandering of coefficients one way or another. Some transformation of the $a_{2,i,j}$ to $\mathbb{Z}$ can be written, as in A276817. A similar transformation certainly exists for $m > 2$. However, up to symmetry, both A111785 and A276817 appear to list duplicate numbers, implying the existence of better definitions. The next section moves on to a plausible refinement, which greatly simplifies all inversion formulae if True.

VI. GENERAL CONJECTURE

We conjecture that the partition triangle for every $m > 0$ can be regularized to a triangle where row length equals row index. The reverse expansion then takes a simple form

$$x = G''_m(\mathbf{c}, y) = y\left(1 + \sum_{i=1}^{\infty}\sum_{j=1}^{i} y^i\,(-m)^{-j}\,a_{m,i,j}\,\widetilde{a}_{i,j}(\mathbf{c})\right), \tag{15}$$

with $c_0 = 1$ for all $m > 0$. A few hours of empirical research at the OEIS reveals conjectural definitions for both $a_{m,i,j}$ and $\widetilde{a}_{i,j}$, which involve only a few special functions.

One formulation of the $a_{m,i,j}$ involves only the multifactorial function

$$(n(!^m)) = \begin{cases} n - m > 1 & n((n-m)(!^m)) \\ n - m \le 1 & n \end{cases}. \tag{16}$$

Then the triangle of coefficients takes a simple form

$$a_{m,i,j} = \frac{(i + 1 + m\,(j-1))(!^m)}{(i+1)(!^m)}; \quad (i \ge 1,\ i \ge j \ge 1). \tag{17}$$

The $\widetilde{a}_{i,j}(\mathbf{c})$ require elimination of dependence on arguments $\mathbf{w}$. To do so, we collect terms of equal $L(\mathbf{w})$. First introduce a function $\{\mathbf{w}\}$, which essentially counts the permutation symmetry of a monomial term via the factorial function. In a simple, direct calculation of $\{\mathbf{w}\}$, tally the like integers in $\mathbf{w}$ and list the tally values in a new collection $\mathbf{v}$, then

$$\{\mathbf{w}(\mathbf{v})\} = \prod_{v_j \in \mathbf{v}} v_j! \tag{18}$$

It's also useful to introduce the Bell polynomials, with defining recursion

$$B_0(\mathbf{c}) = 1; \quad B_{n+1}(\mathbf{c}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(\mathbf{c})\,c_{i+1}. \tag{19}$$

This recursion produces polynomials of the form

$$B_i(\mathbf{c}) = \sum_{j}^{p(i)} B_i(\mathbf{w}_j, \mathbf{c}) = \sum_{j}^{p(i)} b(\mathbf{w}_j)\, \widetilde{a}_i(\mathbf{w}_j, \mathbf{c}). \tag{20}$$

The coefficients $b(\mathbf{w})$ lead to an alternate definition

$$\{\mathbf{w}\} = [\mathbf{w}]/b(\mathbf{w}). \tag{21}$$

In some particular ordering, $\{\mathbf{w}\}$ is given by A096162, $[\mathbf{w}]$ by A036038, and $b(\mathbf{w})$ by A036040.

Finally, the conjectured $\widetilde{a}_{i,j}(\mathbf{c})$ are given as

$$\widetilde{a}_{i,j}(\mathbf{c}) = \sum_{L(\mathbf{w})=j} \frac{\widetilde{a}_i(\mathbf{w}, \mathbf{c})}{\{\mathbf{w}\}}, \tag{22}$$

where the summation goes over all $\mathbf{w}$ of length $L(\mathbf{w}) = j$. The partial Bell polynomials $B_{i,j}$ and the ordinary Bell polynomials $\hat{B}_{i,j}$ are also length ordered,

$$B_{i,j}(\mathbf{c}) = \sum_{L(\mathbf{w})=j} b(\mathbf{w})\, \widetilde{a}_i(\mathbf{w}, \mathbf{c}), \tag{23a}$$

$$\hat{B}_{i,j}(\mathbf{c}) = \sum_{L(\mathbf{w})=j} \hat{b}(\mathbf{w})\, \widetilde{a}_i(\mathbf{w}, \mathbf{c}), \tag{23b}$$

The expansion coefficients relate by the conversion factor

$$1 = \frac{\hat{b}(\mathbf{w})}{b(\mathbf{w})} \left(\frac{1}{\prod_k w_k!} \frac{(\sum_k w_k)!}{L(\mathbf{w})!} \right) = \frac{[\mathbf{w}]}{b(\mathbf{w})} \frac{\hat{b}(\mathbf{w})}{L(\mathbf{w})!}, \tag{24}$$

which readily applies to the polynomial triangle,

$$\widetilde{a}_{i,j}(\mathbf{c}) = \sum_{L(\mathbf{w})=j} \frac{b(\mathbf{w})}{[\mathbf{w}]}\, \widetilde{a}_i(\mathbf{w}, \mathbf{c}) = \sum_{L(\mathbf{w})=j} \frac{\hat{b}(\mathbf{w})}{L(\mathbf{w})!}\, \widetilde{a}_i(\mathbf{w}, \mathbf{c}) = \frac{\hat{B}_{i,j}(\mathbf{c})}{j!}. \tag{25}$$

The summation ends up hidden in the standard definition of the ordinary Bell polynomials.

VII. COMMENTS ON SIMPLIFICATION

The conjectural form separates calculation of the partition triangles for reverse series m into two tasks: listing the coefficients $a_{m,i,j}$ and listing the polynomials $\widetilde{a}_{i,j}(\mathbf{c})$. For any given m the $\widetilde{a}_{i,j}(\mathbf{c})$ are identical. Then all of the m dependence resides in the triangles $a_{m,i,j}$, which are easy to calculate.

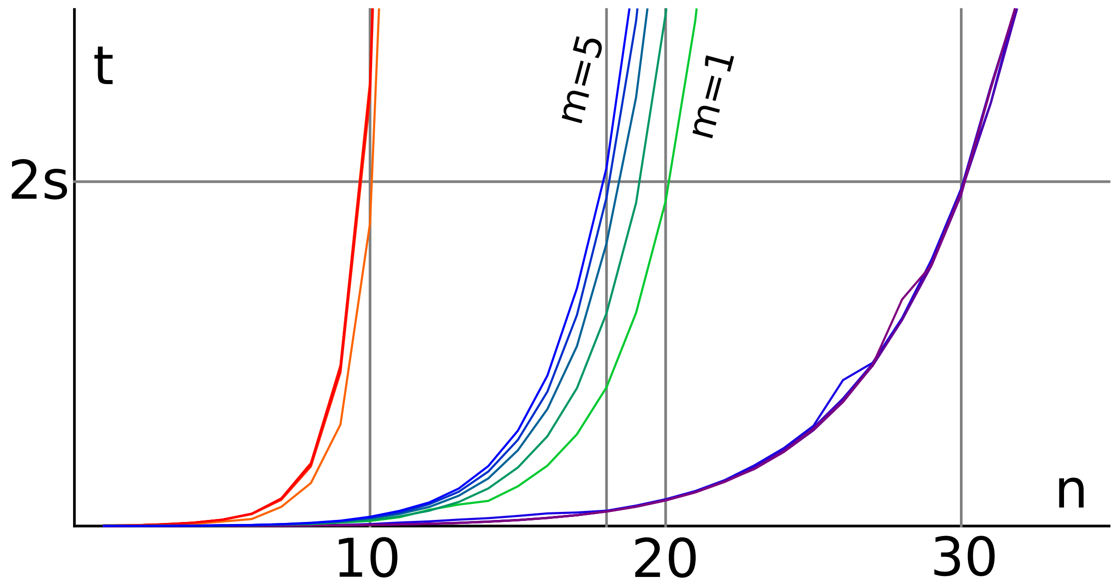

FIG. 1. Comparison of Mathematica algorithm time. Orange: Using `InverseSeries`. Blue-Green: faster algorithm using section IV. Purple: possible fastest algorithm using section VI. The time per n increases extraordinarily as n increases and seems to show the expected $\mathcal{O}(\sum_n p(n))$ complexity.

The majority of computational effort resides in calculation of the $\widetilde{a}_{i,j}(\mathbf{c})$. By row, the complexity of this task is $\mathcal{O}(p(n))$. A Mathematica implementation using Eq. 22, which seems to be the optimal definition, calculates an expansion up to $n = 30$ in about 2 seconds.

If conjectural Eq. 15 is true as appears, it may provide the fastest known method for enumerating the entries of a partition triangle associated with any reverse series m. Figure 1 shows timing tests up to two seconds.

VIII. BRUTE FORCE PROVING

Sections IV and VI describe two distinct algorithms, one conceptually simple and the other a plausible conjecture. Introducing an upper limit to the i summation of Eq. 15, we can enumerate truncated expansions by either method, and test for equivalence. For cases $m = 1, 2$ up to $n = 30$, equivalence is proven. For cases $m = 3, 4, 5$ up to $n = 20$, equivalence is proven. These results strongly suggest that the conjecture is True.

IX. EXAMPLE: LINEAR TERM LEADING

A. Row 5

The solution of case $m = 1$ is determined by the partition triangle A111785, and appears to be solved by the regular triangle A283298 combined with the polynomial triangle $\widetilde{a}_{i,j}(\mathbf{c})$. If the conjecture is true, summation expansions obtained via A111785 and A283298 must match row-by-row. Row 5 provides an interesting case where $p(5) = 7 > 5$.

Up to a sign, the coefficients from A283298 are

$$a_{1,5,j} = \{1, 7, 56, 504, 5040\}, \tag{26}$$

and the polynomials are

$$\widetilde{a}_{5,j}(\mathbf{c}) = \left\{ c_5,\ c_2c_3 + c_1c_4,\ \frac{1}{2}c_1c_2^2 + \frac{1}{2}c_1^2c_3,\ \frac{1}{6}c_1^3c_2,\ \frac{1}{120}c_1^5 \right\}. \tag{27}$$

As required by Eq. 22, the fractional denominator of each monomial coefficient equals to the product of exponent factorials, and in each column the sum of all powers equals to the column index.

Summing over index j obtains the polynomial expansion

$$\sum_j a_{1,5,j}\ (-1)^m\ \widetilde{a}_{5,j}(\mathbf{c}) = -c_5 + 7\ c_4c_1 + 7\ c_3c_2 - 28\ c_3c_1^2 - 28\ c_2^2c_1 + 84\ c_2c_1^3 - 42\ c_1^5. \tag{28}$$

Listing the coefficients of each monomial in the order above,

$$a_{5,j} = \{-1, 7, 7, -28, -28, 84, -42\}. \tag{29}$$

These are exactly the numbers in the fifth row of A111785. At least this is not a counter example to the conjecture.

B. Approximation of $\arcsin(x)$ around $x = 0$

The expansion coefficients of $\sin(x)$ around $x = 0$ are easy to determine, as successive derivatives take a simple form

$$\left.\frac{d^n}{dx^n}\sin(x)\right|_0 = \begin{cases} n \text{ odd} & (-1)^{(n-1)/2} \\ n \text{ even} & 0 \end{cases}. \tag{30}$$

TABLE I. Rationals for calculating arcsin(x) expansion coefficients around $x = 0$.

	$a_{1,i,j}$						$\widetilde{a}_{i,j}$					
$i \diagdown j$	1	2	3	4	5	6	1	2	3	4	5	6
2	1	4	·	·	·	·	$-\frac{1}{6}$	0	·	·	·	·
4	1	6	42	336	·	·	$\frac{1}{120}$	$\frac{1}{72}$	0	0	·	·
6	1	8	72	720	7920	95040	$-\frac{1}{5040}$	$-\frac{1}{720}$	$-\frac{1}{1296}$	0	0	0

Assuming a map $c_n \longleftarrow v_{n+1}$, the expansion coefficients are

$$v_n = \begin{cases} n \text{ odd} & \frac{(-1)^{(n-1)/2}}{n!} \\ n \text{ even} & 0 \end{cases}. \tag{31}$$

Odd rows of the polynomial triangle have all entries zero. Approximation of the inverse function to order x^7 only requires rows $i = 2, 4, 6$, as in Table I. Inserting tabulated values into Eq. 15, with the extra factor $(-1)^{-j}$, obtains the expansion

$$\arcsin(y) = G_1''(y) = y\left(1 + \frac{1}{6}y^2 + \frac{3}{40}y^4 + \frac{5}{112}y^6\right) + \mathcal{O}(y^8). \tag{32}$$

As required, the first four expansion coefficients equal to the values tabulated at OEIS, numerators A055786 and denominators A002595.

X. EXAMPLE: QUADRATIC TERM LEADING

A. Row 5

As with linear case, a plausible quadratic reduction exists between partition triangle A276738, and regular triangle A283247. Again summation expansions must match row-by-row. Up to a sign, the row 5 coefficients from A283247 are

$$a_{2,5,j} = \{1, 8, 80, 960, 13440\}, \tag{33}$$

and the polynomials are as in Eq. 27. Combining by Eq. 15 picks up an extra factor of $(-2)^j$, then

$$\sum_j a_{2,5,j}\,(-2)^m\,\widetilde{a}_{5,j}(\mathbf{c}) = -\frac{1}{2}\,c_5 + 2\,c_4c_1 + 2\,c_3c_2 - 5\,c_3c_1^2 - 5\,c_2^2c_1 + 10\,c_2c_1^3 - \frac{7}{2}\,c_1^5. \tag{34}$$

The coefficients are rational, rather than integer, oh no! The gerrymandering in A276738 involves a Hamiltonian line drawing where the quadratic implicit equation takes the form

$$y^2 = x^2 + \sum_{i=1}^{\infty} 4\, c_i\, x^{2+i}. \tag{35}$$

Then we obtain even integer coefficients for all rows including 5. Dividing out the extra factor of two,

$$\begin{aligned} a_{5,j} &= \tfrac{1}{2}\{-\tfrac{1}{2}(4), 2(4)^2, 2(4)^2, -5(4)^3, -5(4)^3, 10(4)^4, -\tfrac{7}{2}(4)^5\} \\ &= \{-1, 16, 16, -160, -160, 1280, -1792\}. \end{aligned} \tag{36}$$

B. Approximation of $\arccos(x)$ around $x = 1$

TABLE II. Rationals for calculating $\arccos(x)$ expansion coefficients around $x = 1$.

	$a_{1,i,j}$						$\widetilde{a}_{i,j}$					
$i \diagdown j$	1	2	3	4	5	6	1	2	3	4	5	6
2	1	5	·	·	·	·	$-\frac{1}{6}$	0	·	·	·	·
4	1	7	63	693	·	·	$\frac{1}{90}$	$\frac{1}{72}$	0	0	·	·
6	1	9	99	1287	19305	328185	$-\frac{1}{2520}$	$-\frac{1}{540}$	$-\frac{1}{1296}$	0	0	0

As above, the rationals of Table II combined with extra factors $(-2)^{-j}$ determine the expansion

$$\frac{\sqrt{2}}{2}\arccos(1-y^2) = G_1''(y) = y\left(1 + \frac{1}{12}y^2 + \frac{3}{160}y^4 + \frac{5}{896}y^6\right) + \mathcal{O}(y^8). \tag{37}$$

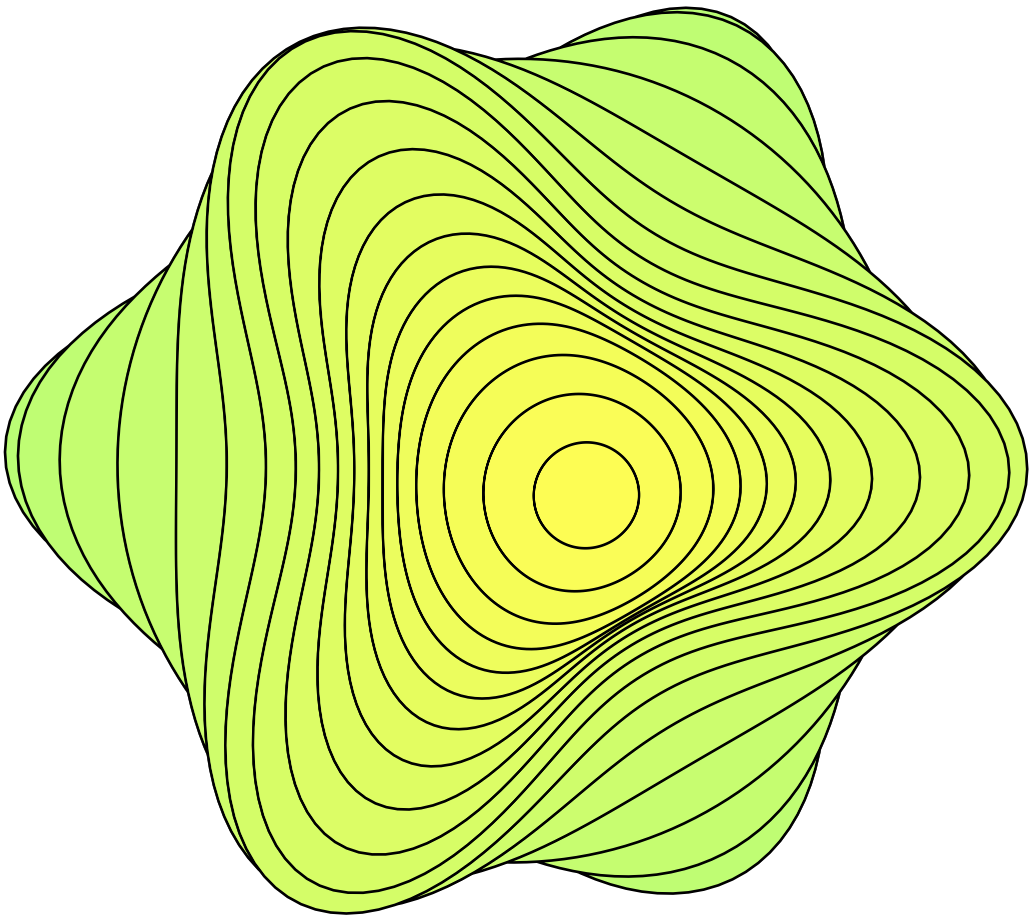

FIG. 2. Yellow-Green Octahedral Surface.

C. Surface Expansions

If the $m = 2$ implicit equation is a multidimensional power series of Cartesian variables,

$$E = F_2(x, y, z) = F_2(r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)) = F_2(r), \tag{38}$$

then a change to spherical coordinates produces an expansion in powers of r, with leading term r^2. The small amplitude limit of the implicit equation becomes the defining relation for a two-dimensional spherical surface, $r = \sqrt{E}$. The reverse series G_2 absorbs the angular dependence into the **c** coefficients. The title page figure "Octahedral Onion" and the yellow-green surface of Fig. 2 show two different projections of a surface that solves a multidimensional implicit equation with octahedral symmetry. Solid lines have constant E and θ, while ϕ varies through a 2π interval.

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